Local uniqueness of certain geodesics related to Heegaard splittings

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ABSTRACT
Suppose \( V_1 \cup_S V_2 \) is a Heegaard splitting and \( D_i \) is an essential separating disk in \( V_i \) such that a component of \( V_i - D_i \) is homeomorphic to \( F_i \times I \), \( i = 1, 2 \). In this paper, we prove that if there is a locally complicated simplicial path in \( C(S) \) connecting \( \partial D_1 \) to \( \partial D_2 \), then the geodesic connecting \( \partial D_1 \) to \( \partial D_2 \) is unique. Moreover, we give a sufficient condition such that \( V_1 \cup_S V_2 \) is keen and the geodesic between any pair of essential disks on the opposite sides has local uniqueness property.

Keywords: Curve complex; subsurface projection; keen Heegaard splitting; geodesic.

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1. Introduction
Let \( M \) be a 3-manifold and \( S \) an embedded closed orientable surface in \( M \). If \( S \) divides \( M \) into two compression bodies \( V \) and \( W \) such that \( S = \partial_+ V = \partial_+ W \), then

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For any essential disks \( g \) genus \( \geq s \) satisfying the disks realizing the Hempel distance are separating. In this paper, we consider the existence of keen Heegaard splittings obtained, see [1, 3, 9, 12–14].

In 2001, Hempel [5] introduced the concept of the distance of a Heegaard splitting obtained, see [5, 6, 8, 11]. In [7], Ido et al. introduced the concept of keen Heegaard splitting. A Heegaard splitting \( V \cup_S W \) is called keen if its Hempel distance is realized by a unique pair of essential disks on the opposite sides of \( S \). Moreover, they proved the existence of the strongly keen Heegaard splitting of genus \( g \) with distance \( n \) for each \( n \geq 2 \) and \( g \geq 3 \). In [2], E proved the existence of keen weakly reducible Heegaard splitting of genus \( g \) with \( g \geq 3 \). In [2], the authors considered the existence of keen Heegaard splittings of closed 3-manifolds and the disks realizing the Hempel distance must be non-separating. In this paper, we consider the existence of keen Heegaard splittings satisfying the disks realizing the Hempel distance are separating.

Let \( M \) be a compact orientable 3-manifold and \( F_1, F_2 \) two components of \( \partial M \) and \( V_1 \cup_S V_2 \) a Heegaard splitting of \( M \) satisfying that \( F_i \) is a component of \( \partial \cdot V_i \) where \( i = 1, 2 \). Let \( D_i \) be an essential separating disk in \( V_i \) such that a component of \( V_i - D_i \) is homeomorphic to \( F_i \times I \) for \( i = 1, 2 \). Denote the component of \( V_i - D_i \) which is not homeomorphic to \( F_i \times I \) by \( V'_i \) and \( S_i = \partial_i(F_i \times I) \cap S \), where \( i = 1, 2 \). Let \( f_i \) be the projection from \( C(S) \) to \( C(\partial_i V'_i) \) determined by \( D_i \) and \( V'_i \) the set of vertices in \( C(S) \) represented by boundaries of essential disks in \( V_i \). Let \( d_i = d_{C(\partial_i V'_i)}(\partial D_i, f_i(\partial D_i)) \), where \( \partial D_i \) denotes the set of vertices in \( C(\partial_i V'_i) \) represented by boundaries of essential disks in \( V'_i \) and \( \{i, j\} = \{1, 2\} \).

**Theorem 1.1.** Suppose \( V_1 \cup_S V_2 \) is a strongly irreducible Heegaard splitting and \( \{\partial D_1 = a_0, a_1, \ldots, a_n = \partial D_2\} \) is a simplicial path in \( C(S) \) connecting \( \partial D_1 \) to \( \partial D_2 \) such that \( a_0 \subset \partial_i V'_i \) and \( a_{n-1} \subset \partial_i V'_i \), If \( \text{diam}_{C(S-a_i)}(\pi_i(a_0), \pi_i(a_{i+1})) \geq M + 7 \) for \( 1 \leq i \leq n - 1 \) and \( d_j \geq M + 1 \) for \( j = 1, 2 \), then

1. \( V_1 \cup_S V_2 \) is a keen Heegaard splitting with distance \( n \).
2. For any essential disks \( D_i \) in \( V_i \) where \( i = 1, 2 \), the geodesic connecting \( \partial D_i \) to \( \partial D_j \) in \( C(S) \) passes through the geodesic \( \{a_1, a_2, \ldots, a_{n-1}\} \).

If we consider a special case for \( n = 2 \), following the lines of the proof of Theorem 1.1 we have the following corollary.

**Corollary 1.2.** Suppose \( V_1 \cup_S V_2 \) is a strongly irreducible Heegaard splitting and \( \{\partial D_1, a, \partial D_2\} \) is a simplicial path connecting \( \partial D_1 \) to \( \partial D_2 \) in \( C(S) \) such that a does
not lie in $S_i$ for $i = 1, 2$. If $d_{C(S-a)}(\partial D_i, \partial D_k) \geq 7$ and $d_i \geq M + 1$ for $i = 1, 2$, then

1. $V_1 \cup S V_2$ is a keen Heegaard splitting.
2. For any essential disks $D^i$ in $V_i$ where $i = 1, 2$, the geodesic connecting $\partial D^1$ to $\partial D^2$ in $C(S)$ passes through $\{a\}$.

The paper is organized as follows. In Sec. 2, we review some necessary preliminaries. The proofs of the main results are given in Sec. 3.

2. Preliminaries

Suppose $F$ is a compact orientable surface of genus at least 1. The curve complex $C(F)$ of $F$, first introduced by Harvey [4], is defined as follows: each vertex is the isotopy class of an essential simple closed curve in $F$ and a set of vertices in $C(F)$ which can be represented by disjoint simple closed curves in $F$ determines a simplex in $C(F)$. When $F$ is a torus or once-punctured torus, the curve complex of $F$, defined by Masur and Minsky [10], is the complex whose vertices are isotopy classes of essential simple closed curves in $F$, and $(k + 1)$ vertices determine a $k$-simplex if they can be realized by curves which mutually intersect in only one point.

For any two vertices $\alpha, \beta$ in $C(F)$, the distance between $\alpha$ and $\beta$, denoted by $d_{C(F)}(\alpha, \beta)$, is defined to be the minimal number of 1-simplices in all possible simplicial paths connecting $\alpha$ to $\beta$. The simplicial path realizes the distance between $\alpha$ and $\beta$ is called a geodesic. Let $A$ and $B$ be any two sets of vertices in $C(F)$. The diameter of $A$, denoted by $\text{diam}_{C(F)}(A)$, is defined to be $\max\{d(x, y) \mid x, y \in A\}$. The distance between $A$ and $B$, denoted by $d_{C(F)}(A, B)$, is defined to be $\min\{d(x, y) \mid x \in A, y \in B\}$. Suppose $V \cup S W$ is a Heegaard splitting of $M$. The distance of $V \cup S W$ is defined to be $d_{C(S)}(D_V, D_W)$, where $D_V (D_W)$ denotes the set of vertices in $C(S)$ which represent boundaries of essential disks in $V (W)$.

Let $F$ be a compact orientable surface of genus at least 1 with non-empty boundary. Denote the arc and curve complex of $F$ by $AC(F)$. Vertices of $AC(F)$ are isotopy classes of essential arcs or curves in $F$ and $(k + 1)$ vertices determine a $k$-simplex if they can be represented by pairwise disjoint arcs or curves. The distance between two vertices $\alpha, \beta$, denoted by $d_{AC(F)}(\alpha, \beta)$, is defined to be the minimal number of 1-simplices in a simplicial path joining $\alpha$ to $\beta$ over all such possible paths.

Let $F'$ be a subsurface of $F$ such that each component of $\partial F'$ is essential in $F$. By the definition of projections to subsurfaces in [10], there is a natural map $\kappa_{F'}$ from vertices of $C(F)$ to finite subsets of vertices of $AC(F')$ defined as follows: For every vertex $[\gamma]$ in $C(F)$, take a curve $\gamma$ in the isotopy class such that $[\gamma \cap F']$ is minimal. If $\gamma \cap F' = \emptyset$, then $\kappa_{F'}([\gamma]) = \emptyset$. If $\gamma \cap F' \neq \emptyset$, then $\kappa_{F'}([\gamma])$ is the union of the isotopy classes of essential components of $\gamma \cap F'$. Furthermore, there is a natural map $\sigma_{F'}$ from vertices of $AC(F')$ to finite subsets of vertices of $C(F')$: For
every vertex $[\beta]$ in $AC(F')$, if $[\beta]$ is the isotopy class of an essential simple closed curve in $F'$, then $\sigma_{F'}([\beta]) = [\beta]$; if $[\beta]$ is the isotopy class of an essential arc, then $\sigma_{F'}([\beta])$ is the union of the isotopy classes of essential boundary components of the regular neighborhood of $[\beta] \cup \partial F'$. Then $\pi_{F'} = \sigma_{F'} \circ \kappa_{F'}$ is a map from vertices of $C(F)$ to finite subsets of vertices of $C(F')$.

For any two vertices $a_1, a_2 \in C(F)$, if $\pi_{F'}(a_i) \neq \emptyset$ for $i = 1, 2$, then $\text{diam}_{C(F')}(\pi_{F'}(a_1), \pi_{F'}(a_2)) \leq 2$ and $\text{diam}_{C(F')}(\pi_{F'}(a_i)) \leq 2$ where $i = 1, 2$.

The following so-called Bounded Geodesic Image Theorem, due to Masur and Minsky, will be used in our discussion.

**Lemma 2.1** ([1]). Let $F'$ be an essential sub-surface of $F$, and $\gamma$ a geodesic segment in $C(F)$, such that $\pi_{F'}(v) \neq \emptyset$ for every vertex $v$ of $\gamma$. Then there is a constant $M$ depending only on $F'$ so that $\text{diam}_{C(F')}(\pi_{F'}(\gamma)) \leq M$.

### 3. Proofs of Main Results

Let $S$ be an orientable closed surface with $g(S) \geq 3$ and $J_1, J_2$ two separating essential simple closed curves in $S$. A simplicial path $\{J_1 = a_0, a_1, \ldots, a_{n-1}, a_n = J_2\}$ connecting $J_1$ to $J_2$ in $C(S)$ is called locally complicated if $a_i$ is non-separating in $S$ and $\text{diam}_{C(S-a_i)}(\pi_i(J_1), \pi_i(a_{i+1})) \geq M + 7$ for $1 \leq i \leq n - 1$ and $n \geq 2$, where $\pi_i$ is the projection from $C(S)$ to $C(S - a_i)$ determined by $a_i$. Denote the component of $S - J_1$ which $a_1$ does not lie in by $S_1$ and the component of $S - J_2$ which $a_{n-1}$ does not lie in by $S_2$. Then we have the following lemma.

**Lemma 3.1.** Suppose $S$ is an orientable closed surface with $g(S) \geq 3$ and $J_1, J_2$ two separating essential simple closed curves in $S$ satisfying that there is a locally complicated simplicial path $\{J_1 = a_0, a_1, \ldots, a_{n-1}, a_n = J_2\}$ connecting $J_1$ to $J_2$ in $C(S)$, then

1. $\{J_1, a_1, \ldots, a_{n-1}, J_2\}$ is the unique geodesic connecting $J_1$ to $J_2$ in $C(S)$.
2. For each essential simple closed curve $C_i \subset S_i$, $\{J_1, a_1, \ldots, a_{n-1}, C_2\}$ and $\{C_1, a_1, \ldots, a_{n-1}, J_2\}$ are the unique geodesics in $C(S)$ connecting $C_i$ to $J_1$, where $\{i, j\} = \{1, 2\}$, and $\{C_1, a_1, \ldots, a_{n-1}, C_2\}$ is the unique geodesic in $C(S)$ connecting $C_1$ to $C_2$.

**Proof.** (1) Suppose $\{J_1 = b_0, b_1, \ldots, b_k = J_2\}$ is a geodesic in $C(S)$ connecting $J_1$ to $J_2$. Since $\{J_1, a_1, \ldots, a_{n-1}, J_2\}$ is a simplicial path connecting $J_1$ to $J_2$, $k \leq n$. Since $a_{n-1} \cap J_2 = \emptyset$, $a_{n-1} \neq b_i$ for $0 \leq i < k - 1$. Since $a_{n-1} \neq J_2$, $\pi_{n-1}(J_2) \neq \emptyset$. If $a_{n-1} \neq b_{k-1}$, then $\pi_{n-1}(b_i) \neq \emptyset$ for $0 \leq i \leq k - 1$. So by Lemma 2.1, $\text{diam}_{C(S-a_{n-1})}(\pi_{n-1}(J_1), \pi_{n-1}(J_2)) \leq M$, a contradiction. So $a_{n-1} = b_{k-1}$.

If $a_{n-2} \neq b_{k-2}$, we can get $\text{diam}_{C(S-a_{n-2})}(\pi_{n-2}(J_1), \pi_{n-2}(a_{n-1})) \leq M$ by a similar argument, a contradiction. So $a_{n-2} = b_{k-2}$.

An induction implies $k = n$ and $a_i = b_i$ for $0 < i < k$. So $\{J_1, a_1, \ldots, a_{n-1}, J_2\}$ is the unique geodesic connecting $J_1$ to $J_2$ in $C(S)$. 


(2) Suppose \( C_1 \) is an essential simple closed curve in \( S_1 \) and \( \{ C_1 = b_0, b_1, \ldots, b_k = J_2 \} \) is a geodesic connecting \( C_1 \) to \( J_2 \) in \( C(S) \).

If \( C_1 \cap J_2 = \emptyset \), then \( \{ J_1, C_1, J_2 \} \) is a simplicial path connecting \( J_1 \) to \( J_2 \) in \( C(S) \). Since \( d_{C(S)}(J_1, J_2) \geq 2 \), both \( \{ J_1, a_1, J_2 \} \) and \( \{ J_1, C_1, J_2 \} \) are geodesics in \( C(S) \) connecting \( J_1 \) to \( J_2 \). Since \( C_1 \subset S_1 \) and \( a_1 \cap S_1 = \emptyset \), \( a_1 \neq C_1 \) and \( \pi_1(C_1) \neq \emptyset \). So \( \text{diam}_{C(S-a_1)}(\pi_1(J_1), \pi_1(J_2)) \leq 4 \), a contradiction. So \( C_1 \cap J_2 \neq \emptyset \) and \( k \geq 2 \).

Since \( a_{n-1} \cap J_2 = \emptyset \), \( a_{n-1} \neq b_i \) for \( 0 \leq i < k - 1 \). If \( a_{n-1} \neq b_{k-1} \), then \( \pi_{n-1}(b_i) \neq \emptyset \) for \( 0 \leq i \leq k \). So \( \text{diam}_{C(S-a_{n-1})}(\pi_{n-1}(C_1), \pi_{n-1}(J_2)) \leq \mathcal{M} \). Since \( C_1 \subset S_1 \), \( C_1 \cap J_1 = \emptyset \). So \( \text{diam}_{C(S-a_{n-1})}(\pi_{n-1}(J_1), \pi_{n-1}(C_1)) \leq 2 \).

Thus
\[
\text{diam}_{C(S-a_{n-1})}(\pi_{n-1}(J_1), \pi_{n-1}(J_2)) \leq \text{diam}_{C(S-a_{n-1})}(\pi_{n-1}(J_1), \pi_{n-1}(C_1)) \\
+ \text{diam}_{C(S-a_{n-1})}(\pi_{n-1}(C_1), \pi_{n-1}(J_2)) \\
\leq \mathcal{M} + 2,
\]
a contradiction. So \( a_{n-1} = b_{k-1} \).

A similar argument as above implies that \( k = n \) and \( a_i = b_i \) for \( 1 \leq i < k \). So \( \{ C_1, a_1, \ldots, a_{n-1}, J_2 \} \) is the unique geodesic connecting \( C_1 \) to \( J_2 \) in \( C(S) \).

For an essential simple closed curve \( C_2 \) in \( S_2 \), with the same method, we can prove that \( \{ J_1, a_1, \ldots, a_{n-1}, C_2 \} \) is the unique geodesic connecting \( J_1 \) to \( C_2 \) in \( C(S) \).

Suppose \( C_1 \) is an essential simple closed curve in \( S_1 \) where \( i = 1, 2 \) and \( \{ C_1 = b_0, b_1, \ldots, b_k = C_2 \} \) is a geodesic connecting \( C_1 \) to \( C_2 \) in \( C(S) \). Suppose \( C_1 = C_2 \).

Since \( d_{C(S)}(J_1, J_2) \geq 2 \), \( \{ J_1, C_2, J_2 \} \) is a geodesic in \( C(S) \) connecting \( J_1 \) to \( J_2 \). Since \( a_{n-1} \neq C_2 \), \( \text{diam}_{C(S-a_{n-1})}(\pi_{n-1}(J_1), \pi_{n-1}(J_2)) \leq \mathcal{M} \), a contradiction. So \( C_1 \neq C_2 \).

If \( C_1 \cap C_2 = \emptyset \), then \( \{ J_1, C_1, C_2, J_2 \} \) is a simplicial path connecting \( J_1 \) to \( J_2 \) in \( C(S) \). So \( n = d_{C(S)}(J_1, J_2) \leq 3 \). Suppose \( n = 3 \), \( \{ J_1, a_1, a_2, J_2 \} \) is the unique geodesic in \( C(S) \) connecting \( J_1 \) to \( J_2 \). Since \( C_1 \cap J_1 = \emptyset \), \( a_2 \neq C_1 \). So \( \pi_2(C_1) \neq \emptyset \) for \( i = 1, 2 \) and \( \text{diam}_{C(S-a_2)}(\pi_2(J_1), \pi_2(J_2)) \leq 6 \), a contradiction.

So \( n = 2 \) and \( \{ J_1, a_1, J_2 \} \) is the unique geodesic in \( C(S) \) connecting \( J_1 \) to \( J_2 \).

In this case, \( a_1 \cap S_1 = \emptyset \), so \( a_1 \neq C_1 \) for \( i = 1, 2 \). Thus \( \pi_1(C_1) \neq \emptyset \) for \( i = 1, 2 \). So \( \text{diam}_{C(S-a_1)}(\pi_1(J_1), \pi_1(J_2)) \leq 6 \), a contradiction. Thus \( C_1 \cap C_2 \neq \emptyset \).

Since \( a_{n-1} \cap C_2 = \emptyset \), \( a_{n-1} \neq b_i \) for \( 0 \leq i < k - 1 \). If \( a_{n-1} \neq b_{k-1} \), then \( \pi_{n-1}(b_i) \neq \emptyset \) for \( 0 \leq i \leq k \). So \( \text{diam}_{C(S-a_{n-1})}(\pi_{n-1}(C_1), \pi_{n-1}(C_2)) \leq \mathcal{M} \). Since \( C_1 \cap J_1 = \emptyset \), \( \text{diam}_{C(S-a_{n-1})}(\pi_{n-1}(J_1), \pi_{n-1}(C_1)) \leq 2 \) where \( i = 1, 2 \).

Thus
\[
\text{diam}_{C(S-a_{n-1})}(\pi_{n-1}(J_1), \pi_{n-1}(J_2)) \leq \text{diam}_{C(S-a_{n-1})}(\pi_{n-1}(J_1), \pi_{n-1}(C_1)) \\
+ \text{diam}_{C(S-a_{n-1})}(\pi_{n-1}(C_1), \pi_{n-1}(J_2)) \\
+ \text{diam}_{C(S-a_{n-1})}(\pi_{n-1}(J_2), \pi_{n-1}(C_2)) \\
\leq \mathcal{M} + 4,
\]
a contradiction. So \( a_{n-1} = b_{k-1} \).
A similar argument as above implies that $k = n$ and $a_i = b_i$ for $1 \leq i < k$. So \{\C_1, a_1, \ldots, a_{n-1}, \C_2\} is the unique geodesic connecting $\C_1$ to $\C_2$ in $C(S)$.

This completes the proof of the lemma.

Before proving the results, let's recall some notations. Let $M$ be a compact orientable 3-manifold and $F_1, F_2$ two components of $\partial M$ and $V_1 \cup_S V_2$ a Heegaard splitting of $M$ satisfying that $F_i$ is a component of $\partial_v V_i$ where $i = 1, 2$. Let $D_i$ be an essential separating disks in $V_i$ such that a component of $V_i - D_i$ is homeomorphic to $F_i \times I$ for $i = 1, 2$. Denote the component of $V_i - D_i$ which is not homeomorphic to $F_i \times I$ by $V_i'$ and $S_i = \partial_+ (F_i \times I) \cap S$, where $i = 1, 2$. Let $f_i$ be the projection from $C(S)$ to $C(\partial_+ V_i^\prime)$ determined by $D_i$ and $V_i$, the set of vertices in $C(S)$ represented by boundaries of essential disks in $V_i$. Let $d_i = d_{C(\partial_+ V_i^\prime)}(D_{V_i^\prime}, f_i(D_{V_i}))$, where $D_{V_i^\prime}$ the set of vertices in $C(\partial_+ V_i^\prime)$ represented by boundaries of essential disks in $V_i'$ and \( \{i, j\} = \{1, 2\} \).

We are now equipped to prove Theorem 1.1.

**Proof.** (1) Suppose $V_1 \cup_S V_2$ is not keen with distance $n$. Then there is another pair of essential disks $D_1'$ in $V_1$ and $D_2'$ in $V_2$ such that $d_{C(S)}(\partial D_1', \partial D_2') \leq n$. Suppose $\{\partial D_1' = c_0, c_1, \ldots, c_k = \partial D_2'\}$ is a geodesic in $C(S)$ connecting $\partial D_1'$ to $\partial D_2'$ and $k \leq n$.

**Claim 1.** $D_i' \neq D_i$ for $i = 1, 2$.

Otherwise, suppose $D_1' = D_1$. Since $\{D_1', D_2'\}$ is a distinct pair of disks from $\{D_1, D_2\}$, $D_2' \neq D_2$. Isotope $D_2'$ such that $|D_2' \cap D_2| = 0$. If $D_2' \cap D_2 = \emptyset$, then $D_2'$ is an essential disk in $V_2'$ and $f_2(\partial D_2') = \partial D_2'$.

Suppose $D_2' \cap D_2 \neq \emptyset$. Since $V_2$ is irreducible, an innermost loop argument implies that there is no closed curve component in $D_2' \cap D_2$. Choose an arc $\gamma$ from $D_2' \cap D_2$ such that $\gamma$ is outermost in $D_2'$. Then $\gamma$ cuts a disk $D_\gamma$ from $D_2'$ such that $D_\gamma \cap D_2 = \gamma$. Suppose $\gamma$ cuts $D_2$ into two components $D_\gamma'$ and $D_\gamma''$. Let $D_\gamma' = D_2' \cup_D \gamma$. Since $|D_\gamma' \cap D_2| = 0$, both $D_\gamma'$ and $D_\gamma''$ are essential disks in $V_2'$ and $d_{C(S)}(\partial D_\gamma', \partial D_\gamma'') \leq n$. So there always exists an essential disk $E$ in $V_2'$ such that $\partial E \in f_2(\partial D_\gamma')$.

If $f_2(c_i) \neq \emptyset$ for $0 \leq i \leq k - 1$, then by Lemma 2.1
\[
\text{diam}_{C(\partial_+ V_2)}(f_2(\partial D_1), f_2(\partial D_2')) \leq \mathcal{M}.
\]

Thus
\[
d_2 = d_{C(\partial_+ V_2)}(f_2(\partial V_1'), \partial V_2') \\
\leq d_{C(\partial_+ V_2)}(f_2(\partial D_1), \partial E) \\
\leq \text{diam}_{C(\partial_+ V_2)}(f_2(\partial D_1), f_2(\partial D_2')) \\
\leq \mathcal{M},
\]
a contradiction. So there exists some $j$ such that $f_2(c_j) = \emptyset$ where $0 \leq j < k$. 


So \( \partial D_1 = c_0, c_1, \ldots, c_j, \partial D_2 \) is a simplicial path in \( C(S) \) connecting \( \partial D_1 \) to \( \partial D_2 \) with length \( j + 1 \). Since \( j < k \leq n, j + 1 \leq n \). Since \( d_{C(S)}(\partial D_1, \partial D_2) = n, j + 1 = n \) and \( \{ \partial D_1 = c_0, c_1, \ldots, c_j, \partial D_2 \} \) is a geodesic in \( C(S) \) connecting \( \partial D_1 \) to \( \partial D_2 \). Since \( c_j \subset S_2 \) and \( a_{n-1} \cap S_2 = \emptyset \), \( c_j \neq a_{n-1} \) and \( \{ \partial D_1 = c_0, c_1, \ldots, c_j, \partial D_2 \} \) is another geodesic different from \( \{ \partial D_1 = a_0, a_1, \ldots, a_n = \partial D_2 \} \). By Lemma 3.1 there is a unique geodesic connecting \( \partial D_1 \) to \( \partial D_2 \), a contradiction. So \( D'_1 \neq D_1 \).

A similar argument implies \( D'_2 \neq D_2 \). This completes the proof of the Claim.

So \( D'_i \neq D_i \) for \( i = 1, 2 \) and \( d_{C(S)}(\partial D_i, \partial D'_i) > n \) where \( \{ i, j \} = \{ 1, 2 \} \). Isotope \( D'_i \) such that \( |D'_i \cap D_i| \) is minimal. If \( D'_i \cap D_i = \emptyset \), then \( D'_i \) is a compressing disk in \( V'_i \) and \( f_i(\partial D'_i) = \partial D'_i \).

Suppose \( D'_1 \cap D_1 \neq \emptyset. \) Since \( V_2 \) is irreducible, an innermost loop argument implies that there is no closed curve component in \( D'_2 \cap D_1 \). Choose an arc \( \alpha \) from \( D'_1 \cap D_1 \) such that \( \alpha \) is outermost in \( D'_1 \). Then \( \alpha \) cuts a disk \( D_0 \) from \( D'_1 \) such that \( D_0 \cap D_1 = \alpha \). Suppose \( \alpha \) cuts \( D_1 \) into two components \( D_1^+ \) and \( D_1^- \). Let \( D'_1 = D'_1 \cup_\alpha D_0 \). Since \( |D'_1 \cap D_1| \) is minimal, both \( D_1^+ \) and \( D_1^- \) are essential disks in \( V'_1 \) and \( \partial D'_1 \in f_1(\partial D'_1) \) where \( i = 1, 2 \). So there always exists an essential disk \( E' \) in \( V'_1 \) such that \( \partial E' \in f_1(\partial D'_1) \).

Since \( d_{C(S)}(\partial D_1, \partial D'_2) > n \geq 2, f_1(\partial D'_2) \neq \emptyset \). If \( f_1(c_j) \neq \emptyset \) for \( 1 \leq i \leq k - 1 \), then by Lemma 2.2 \( \text{diam}_{C(S)}(f_1(\partial D'_1), f_1(\partial D'_2)) \leq M \). Thus

\[
\text{diam}_{C(S)}(f_1(\partial D'_1), f_1(\partial D'_2)) \leq M,
\]

a contradiction. So there exists some \( j \) such that \( f_1(c_j) = \emptyset \) where \( 0 < j < k \).

Since \( f_1(c_j) = \emptyset \), then we can isotope \( c_j \) such that \( c_j \subset S_1 \) and \( c_j \cap \partial D_1 = \emptyset \). So \( \{ \partial D_1, c_1, \ldots, c_{k-1}, \partial D_2 \} \) is a simplicial path in \( C(S) \) connecting \( \partial D_1 \) to \( \partial D_2 \) with length \( k - j + 1 \). Since \( j \geq 1, d_{C(S)}(\partial D_1, \partial D_2) \leq k - j + 1 \leq k \leq n \), a contradiction. Thus \( d_{C(S)}(\partial D'_1, \partial D'_2) > n \) and \( V_1 \cup S V_2 \) is keen with distance \( n \).

This completes the proof of the first part.

(2) Suppose \( \{ \partial D^1 = b_0, b_1, \ldots, b_m = \partial D^2 \} \) is a geodesic connecting \( \partial D^1 \) to \( \partial D^2 \) in \( C(S) \). A similar argument as above implies that there exist \( k_1, k_2 \) such that \( b_{k_i} \subset S_i \) where \( i = 1, 2 \). By Lemma 3.1 \( \{ b_{k_1}, a_1, \ldots, a_{n-1}, b_{k_2} \} \) is the unique geodesic connecting \( b_{k_1} \) to \( b_{k_2} \) in \( C(S) \).

If \( k_1 = k_2 \), since \( b_{k_1} \cap \partial D_1 = \emptyset \) and \( b_{k_2} \cap \partial D_2 = \emptyset \), \( \{ \partial D_1, b_{k_1} = b_{k_2}, \partial D_2 \} \) is a simplicial path connecting \( \partial D_1 \) to \( \partial D_2 \). Thus \( d_{C(S)}(\partial D_1, \partial D_2) = n = 2. \) Since \( a_1 \neq b_{k_1}, \{ \partial D_1, a_1, \partial D_2 \} \) is another geodesic connecting \( \partial D_1 \) to \( \partial D_2 \), a contradiction. So \( k_1 \neq k_2 \).

If \( k_1 < k_2 \), then \( \{ b_{k_1}, b_{k_1+1}, \ldots, b_{k_2} \} \) is a geodesic connecting \( b_{k_1} \) to \( b_{k_2} \) in \( C(S) \). So \( b_{k_1+i} = a_i \) for \( 1 \leq i \leq n - 1 \). Thus the geodesic \( \{ \partial D^1 = b_0, b_1, \ldots, b_m = \partial D^2 \} \)
contains \( \{b_1, a_1, \ldots, a_{n-1}, b_n\} \) as a part and the conclusion holds in this case.

Suppose \( k_2 < k_1 \), then \( \{b_{k_2}, b_{k_2+1}, \ldots, b_{k_1}\} \) is a geodesic connecting \( b_{k_2} \) to \( b_{k_1} \) in \( C(S) \). So \( b_{k_2+i} = a_{n-i} \) for \( 1 \leq i \leq n - 1 \).

Suppose \( f_1(b_{k_2}) = \emptyset \). Then \( b_{k_2} \subset S_1 \) and \( d_{C(S)}(b_{k_1}, b_{k_2}) \leq 2 \). If \( n \geq 3 \), then \( d_{C(S)}(b_{k_1}, b_{k_2}) = n \geq 3 \), a contradiction.

If \( n = 2 \), then \( \{b_{k_2}, a_1, b_{k_1}\} \) is the unique geodesic connecting \( b_{k_2} \) to \( b_{k_1} \). In this case, \( b_{k_1} \subset S_1 \) for \( i = 1, 2 \) and \( a_1 \subset S_1 \). It is obvious that there is another essential simple closed curve \( C \) in \( S - S_1 \) such that \( C \neq a_1 \) and \( \{b_{k_2}, C, b_{k_1}\} \) is another geodesic connecting \( b_{k_2} \) to \( b_{k_1} \) in \( C(S) \), a contradiction. So \( f_1(b_{k_2}) \neq \emptyset \).

Suppose there exists some \( i \) such that \( 0 \leq i < k_2 \) and \( f_1(b_i) = \emptyset \). Then \( b_i \subset S_1 \) and \( b_i \cap a_1 = \emptyset \). This contradicts to \( d_{C(S)}(b_i, a_1) > 2 \). So \( f_1(b_i) \neq \emptyset \) for \( 0 \leq i \leq k_2 \).

Since \( b_k \subset S_2 \), \( b_k \cap \partial D_2 = \emptyset \). So \( \partial D^1, b_1, \ldots, b_k, \partial D_2 \) is a simplicial path connecting \( \partial D^1 \) to \( \partial D_2 \) with length \( k_2 + 1 \).

If \( d_{C(S)}(\partial D^1, \partial D_2) = k_2 + 1 \), then \( \partial D^1, b_1, \ldots, b_k, \partial D_2 \) is a geodesic connecting \( \partial D^1 \) to \( \partial D_2 \). Since \( f_1(b_i) \neq \emptyset \) for \( 0 \leq i \leq k_2 \) and \( f_1(\partial D_2) \neq \emptyset \), a similar argument as above implies \( d_1 \leq M \), a contradiction.

So \( d_{C(S)}(\partial D^1, \partial D_2) < k_2 + 1 \). Suppose \( d_{C(S)}(\partial D^1, \partial D_2) = l \leq k_2 \) and there is a geodesic \( \partial D^1, c_1, \ldots, c_{l-1}, \partial D_2 \) in \( C(S) \) connecting \( \partial D^1 \) to \( \partial D_2 \).

Since \( \partial D_2 \cap a_{n-1} = \emptyset \), then

\[
\{\partial D^1, c_1, \ldots, c_{l-1}, \partial D_2, b_{k_2+1}, \ldots, b_{n-1}, \partial D^2\}
\]

is a simplicial path connecting \( \partial D^1 \) to \( \partial D^2 \) in \( C(S) \) with length \( m - k_2 + l \). Since \( d_{C(S)}(\partial D^1, \partial D^2) = m, m \leq m - k_2 + l \). So \( l \geq k_2 \). Thus \( l = k_2 \) and \( \{\partial D^1, c_1, \ldots, c_{k_2-1}, \partial D_2, b_{k_2+1}, \ldots, \partial D^2\} \) is a geodesic connecting \( \partial D^1 \) to \( \partial D^2 \) in \( C(S) \).

By a similar argument as above we can get that

\[
\text{diam}_{C(S)}(f_1(\partial D^1), f_1(\partial D_2)) \leq M,
\]

and then we have \( d_1 \leq M \), a contradiction. So \( k_2 > k_1 \).

This completes the proof of the theorem.

\[ \square \]

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**References**

Local uniqueness of certain geodesics


